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NON-LOCAL THEORY OF GRANULAR MEDIA

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14. Abstract

A comprehensive theoretical derivation for the equations of motion of granular media in three dimensions. The derivation is based on both the probabilistic theory of motion of discreet particles, and utilizes the concepts of non-local thermodynamics.

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Granular Materials, Non-Local Thermodynamics

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SECTION 1

INTRODUCTION

In a previous work [3] we developed a non-local thermodynamic theory for the thermomechanical behavior of a granular continuum in terms of the concepts of configurational temperature and entropy. The theory was applied to the motion of granular media under surface tractions and constant configurational temperature.

The non-local theory differs from its local counterpart in that the set of internal variables of the local theory is replaced by an internal variable field whose domain is the interior of the material body. The evolution equation for the internal variables of the local theory is replaced with a partial differential equation for the internal field. This equation must be solved in the presence of the prevailing boundary conditions.

In applying the theory to a granular medium certain experimental observations had to be considered. During a loading history the applied load is supported by a sub-set of grains that form a stationary skeleton. This phenomenon is known as "arching". The skeleton is elastically deformable to the extent of the elastic deformability of the individual grains. The stationary particles are called the elastic phase.

The remaining particles are mobile and hence they are called the mobile phase. The mobile particles move through the elastic skeleton in a dissipative fashion and the displacement field of the mobile phase is, in fact, the internal variable field of the non-local thermodynamic theory.

The field equation for the internal variable field is a "diffusion equation" with the exception that the time coordinate is replaced by an intrinsic time, very much along the lines of endochronic plasticity. The solution of this equation gives the displacement field of the mobile phase. This, and the elastic displacement field constitute the total displacement field of the granular continuum. The stress field is thereby determined.

During unloading, however, the entire phase is stationary and the response is elastic in consequence of the deformability of the individual grains. Thus it is apparent that "mobility" i.e., the ratio of the mobile particles to the total number thereof, in a material region, depends on the loading history and may, in fact, depend on the history of the displacement field itself. The manner in which mobility enters more generally as a constitutive variable is a subject for further research.

Summary of Work.

In Section 2 we lay the physical foundations of the theory and develop the concepts of non-local thermodynamics. Appropriate field equations for the displacement fields of the elastic and mobile phases are derived together with appertaining boundary conditions necessary for the solution of the field equations. Particle mobility appears implicitly in the constitutive formulation but is assumed uniform in the material domain in question. It does, however, vary with the loading history. In a more general formulation, mobility itself may be a field with an appropriate field equation. Such a theory will be the basis of future work.

Section 3 is devoted to the application of the theory to a granular domain in the form of a cuboid in compression with its base rigidly fixed under conditions of uniaxial strain. The resulting diffusion equation is solved under conditions whereby the intrinsic time is proportional to the displacement of the top (mobile) surface of the cuboid.

The mobile displacement field is thus determined at all points in the cuboid. Comparison with experiment is made on the plane near the halfway point on the height coordinate of the cuboid. Agreement with experiment is favorable.

The solution was determined by two different techniques. In the first instance the diffusion equation was solved by a finite difference method using first three and then five equidistant nodes along the height of the cuboid. In the second instance, an exact solution was found by assuming the height of the cuboid to be asymptotically large. This assumption was justified

on the basis of the solution obtained by the first method. It was found that the two methods gave results which were very close.

The analytical ("exact") solution was then used to determine the stress as a function of the displacement of the top surface. The calculated result was then compared with experimental data obtained by Gill [4,5] for <u>three</u> sequential loading and unloading histories. Favorable agreement with experiment was obtained.

In Section 4 we deal with the determination of material functions and constants. Experiments are defined whereby these may be determined in one-dimensional as well as three-dimensional cases, on the basis of physically realistic assumptions. The details of this method are discussed in the body of the work.

In Section 5 we give a conclusive discussion of the results and offer recommendations for future research work.

SECTION 2

THEORY OF GRANULAR MOTION

KINEMATICS

The motion of particles in a granular domain is exceedingly complex. Computer visualization of grain motion [5] as well as the experimental observations of Gill [6], and Mogami [7], indicate that there is a subset of particles that move little, if at all, in the course of deformation of a confined granular domain. This set of particles forms a stationary elastic skeleton, relative to which the remainder of the particles move in the course of the macroscopic deformation. It is this motion that is responsible for the dissipation associated with the deformation of granular domains.

We conclude that, in general, particle motion in a granular domain is described in terms of an elastic and mobile phase. The elastic phase (the skeleton) undergoes affine deformation while the motion of the mobile phase is non-affine and is the cause of energy dissipation in the domain. A set of particles is said to undergo affine motion if the "shell" formed by the immediate neighbors of a particle is never traversed by the particle during the motion. Thus, the deformation process of an aggregate neighborhood may be thought to consist of the deformation of the elastic "skeleton" through which mobile particles flow in a diffusive fashion. This view is in accord with the observed phenomenon of "arching" *i.e.*, the formation of skeletal structures that support the applied stress without inhibiting the flow of the granular medium as a whole.

It is observed in aggregates under compression, that the overall deformation of the domain is not a continuous nor a homogeneous process, but consists of stochastic motion of individual particles. At any time only a subset of particles partakes in the aggregate motion. Thus, excluding the (elastic) deformation of individual grains, at any time, the mobile particles move relative to the bulk of the aggregate which is essentially at rest.

In a statistical average sense, a mobile particle moves through a frictional milieu. It is the resistance to this relative motion that gives rise to dissipation in the course of the deformation of aggregate media. Therefore, it appears that the modeling of aggregate behavior hinges on a realistic description of the diffusive motion of mobile particles through an aggregate domain.

To quantify the above ideas, consider the motion of particles in a material neighborhood, ideally an infinitesimal region. Let N be the total number of particles, n the number of mobile particles and u_i^m the (mean) displacement of the mobile particles in a material neighborhood. Also let $\hat{u}_i$ be the mean displacement of the neighborhood and let the operator δ denote the variation of a quantity it precedes. Then:

$$\delta\hat{u}_i = (n / N)\, \delta u_i^m = \delta u_i^p \quad (n / N) \tag{2.1}$$

where we have set $u_i^m \equiv u_i^p$. The "plastic" connotation in δu_i^p may not be the optimal terminology because the mobile displacement is due to translational motion as well as the deformability of the grains. However so long as this is remembered there should be no cause for confusion.

Digression. In reality the stationary phase is elastically deformable with a displacement field u_i^e. If this field were taken into account, then the variation in the mean displacement of the neighborhood would be given by Eq. (2.1a):

$$\delta\hat{u}_i = \{1-(n / N)\}\, \delta u_i^e + (n / N)\, \delta u_i^m \equiv \delta u_i \tag{2.1a}$$

The most important aspect of Eq. (2.1a) is that the mean displacement field is not a mere sum of its elastic and plastic counterparts, as in metal plasticity, but depends strongly on the mobility of the grains. This is a new and important element that must be considered in the synthesis of constitutive equations for granular media.

In the above formulation, account is made of the fact that n is not constant during deformation. At the present time, however, we have limited experimental data regarding the evolution of n. In fact, its measurement is an interesting project for the future. We do know, however, that during compressive loading n is of the order of 0.8 N. This and the fact that $||\delta u_i^e||$ $<< ||\delta u_i^p||$ (where double bars denote the norm) make the first term on the right-hand side of Eq. (2.1a) ignorable relative to the second, so that during compressive loading:

$$\delta u_i \doteq \delta u_i^p \tag{2.1b}$$

A different argument applies during unloading, however. In this case n = 0, *i.e.*, the particles are immobile and deformation of the aggregate is due to the elastic expansion of the the grains. Thus in the course of unloading:

$$\delta u_i = \delta u_i^e \tag{2.1c}$$

The displacement of the mobile particles relative to the neighborhood is then u_i^* where,

$$\delta u_i^* = \delta u_i^m - \delta \hat{u}_i = (1 - n / N)\, \delta u_i^m \tag{2.2}$$

As we shall see u_i^* is important in the determination of the resistive force in course of particle motion. It follows from Eqs. (2.1) and (2.2) that:

$$\delta u_i^* = (1 - n / N) \delta u_i^p \tag{2.2a}$$

THERMODYNAMICS

At present, irreversible thermodynamics deals with the thermodynamic state of a neighborhood. Heredity is dealt with in terms of locally homogeneous neighborhoods whose thermodynamic state is specified by the (locally uniform) strain, and a requisite number of internal variables which are also locally uniform. This approach leads to local constitutive

theories according to which uniform boundary tractions give rise to uniform deformation gradients. However, a different and more realistic point of view is possible if one regards the entire material domain as a thermodynamic system, rather than attempt to infer its behavior from that of an idealized (locally uniform) material subdomain.

Thermodynamics of the Mobile Phase

We begin with a review of thermodynamics of internal variables under isothermal conditions. Let x_i denote the "observable" kinematic variables and X_i the corresponding dual forces. Also let q_i denote the internal variables and Q_i the dual internal forces. If $\psi(x_i,q_i)$ is the Helmholtz free energy of the system, then:

$$X_i = \partial\psi / \partial x_i|_q \tag{2.3a}$$

$$Q_i = -\partial\psi / \partial q_i|_x \tag{2.3b}$$

In the case of linear thermodynamics, following Biot [7], Q_i are related to the rates of change of q_i linearly, through an "Onsager" positive definite and symmetric matrix b_{ij} so that,

$$Q_i = b_{ij}\, \partial q_j / \partial z \tag{2.4}$$

where z is an intrinsic time scale, not necessarily Newtonian. The concept of intrinsic time was first used by Valanis [9,10] in the development of Endochronic plasticity. Equations (2.3b) and (2.4) combine to give the evolution equations for q_i in the form:

$$\partial\psi / \partial q_i + b_{ij}\, \partial q_j / \partial z = 0 \tag{2.5}$$

We note that Eqs. (2.3a) and (2.3b) give rise to the relation:

$$\Sigma\, X_i\, \delta x_i - \Sigma\, Q_i\, \delta q_i = \delta\psi \tag{2.6}$$

where δ denotes an arbitrarily small variation in the quantities that it precedes.

In the past, constitutive theory was developed on the basis of local thermodynamics of a vanishingly small neighborhood in which event x_i denotes the components of the strain tensor of the neighborhood while X_i denotes the components of the dual stress tensor in the sense that,

$$X_i \, dx_i = dW \tag{2.7}$$

where dW is the increment of work per unit volume.

Global Thermodynamics of the Mobile Phase

In the case of global thermodynamics developed here, the entire material domain constitutes the thermodynamic system. The observable variables are the surface displacements U_i^p while the internal variables are the displacements u_i^p in the interior of the domain. The Helmholtz free energy Ψ of the system is the integral of the free energy density ψ over the material domain D. In turn ψ is a function of the gradient of the displacement field u_i^p just like in the case of the local theory.

Thus, insofar as the mobile phase is concerned,

$$\psi = \psi\left(u_{i}^{p},_{j}\right) \tag{2.8}$$

$$\Psi = \int_D \psi\left(u_{i}^{p},_{j}\right) dV \tag{2.9}$$

In the case of the present thermodynamics system Eq. (2.6) becomes:

$$\int_S T_i^p \, \delta u_i^p \, dS - \int_D Q_i \, \delta u_i^p \, dV = \delta\Psi \tag{2.10}$$

where T_i^P are the surface tractions on the mobile phase.

We remark that $\Sigma X_i \delta x_i$ in Eq. (2.6) is an increment of work done by the external forces so that if a body force field were present, then Eq. (2.10) would be modified to include the effect of such a field in which case it would read:

$$\delta\Psi = \int_S T_i^P \, \delta u_i^P \, dS + \int_D f_i^P \, \delta u_i^P \, dV - \int_D Q_i \, \delta u_i^P \, dV \tag{2.10a}$$

In the case of the mobile phase the only possible such field is the gravitational field which we ignore. Thus $f_i^P = 0$.

The evolution equation for the internal variables now becomes:

$$Q_i = b_{ij} \, \partial u_i^P / \partial z \tag{2.11}$$

To obtain the resulting thermodynamic equations we need to derive an explicit expression for $\delta\Psi$. It follows from Eq. (2.9) that,

$$\begin{aligned} \delta\Psi &= \int_D \left(\partial\psi / \partial u_i^P{}_{,j}\right) \delta u_i^P{}_{,j} \, dV = \\ &\int_D \left\{\left(\partial\psi / \partial u_i^P{}_{,j}\right) \delta u_i^P\right\}_{,j} dV + \int_D \left(\partial\psi / \partial u_i^P{}_{,j}\right) \delta u_i^P \, dV \end{aligned} \tag{2.12}$$

When the Green Gauss theorem is applied to the first integral on the right-hand side of Eq. (2.12) and use is made of Eq. (2.1), the following relation results:

$$\int_S \left\{T_i - \left(\partial\psi / \partial u_i^P{}_{,j}\right) n_j\right\} \delta U_i^P \, dS - \int_D \left\{Q_i - \left(\partial\psi / \partial u_i^P{}_{,j}\right)_{,j}\right\} \delta u_i^P dV = 0 \quad (2.13)$$

where use was made of the fact that on the surface $u_i^P = U_i^P$. Since the variations δU_i^P and δu_i^P are arbitrary, the integrands in Eq. (2.13) must vanish separately, in which case the following thermodynamic relations result:

$$T_i = \left\{\partial\psi / \partial u_i^P{}_{,j}\right\} n_j \qquad \text{on S} \qquad (2.14)$$

$$Q_i = \left\{\partial\psi / \partial u_i^P{}_{,j}\right\}_{,j} \qquad \text{in D} \qquad (2.15)$$

Thus using Eq. (2.11) one obtains Eq. (2.16) which is the evolution equation for the internal variables, in the form of a <u>partial differential equation</u> in the spatial coordinates y_i and the intrinsic time z:

$$\left\{\partial\psi / \partial u_i^P{}_{,j}\right\}_{,j} = b_{ij} \, \partial u_j^P / \partial z \quad . \qquad (2.16)$$

In the event that the material is isotropic (in a continuum sense) then $\underset{\sim}{b}$ is an isotropic second order tensor, and in the case where it is a constant, it is proportional to the unit matrix. In this study we take the position that $\underset{\sim}{b}$ is given by Eq. (2.17).

$$b_{ij} = b \, \delta_{ij} \qquad (2.17)$$

where b is a scalar invariant. Equation (2.16) becomes,

$$\left\{\partial\psi / \partial u_i^P{}_{,j}\right\}_{,j} = b \, \partial u_i^P / \partial z \qquad (2.18)$$

Equation (2.18) is the evolution equation for the <u>internal variable field</u> u_i^P. It is a field (diffusion) equation which must be solved in the light of the prevailing displacement or stress boundary conditions and as such it is a

GLOBAL equation Note that in local thermodynamics the evolution equation for the internal variables is independent of the boundary conditions of the material domain.

The Resistance Coefficient b. Physically, resistance is generated by differential particle motion, in our case defined by u_i^* in Eq. (2.2). Thus the resistance force Q_i is given by Eq. (2.19).

$$Q_i = b\, \partial u_i^* / \partial z \qquad (2.19)$$

where b is a material property which depends on the mobility of the grains n/N, their geometry and the nature of the grain surface but also on the confining pressure.

In the non-local theory the lateral stress, brought about by lateral confinement, varies with the height of the cuboid. This poses difficulties in solving the prevailing equations analytically. Analytical solutions are necessary, if the experimental data are to be utilized to obtain the material constants and functions. To this end we have set b to depend on the mean pressure of the entire domain of the cuboid, and therefore on the applied stress σ. In addition b will depend on the porosity w and therefore on the volumetric strain. This, in the uniaxial case, is one third of the axial strain.

Remark. Thought was given to the question of how porosity depends on the strain of a granular neighborhood. Evidently it depends on the positional configuration of the grains as well as their deformation. For, if the external boundary of granular domain were held fixed and the grains were compressed elastically, the porosity would obviously increase. In a loading-unloading sequence the porosity would, therefore, depend on the cumulative strain of the neighborhood.

Because pressure and porosity dependence are due to two independent physical processes, b will depend on these in a multiplicative fashion. Hence:

$$b = b' f(\sigma)\, g(w) \qquad (2.20)$$

The constant b' is a property of the granular medium. We now reason that Q_i is proportional to the number of friction surfaces and, therefore, proportional to n. Hence:

$$b' = nb_o \tag{2.20a}$$

Thus in view of Eqs. (2.19), (2.20) and (2a), the dissipative force Q_i will be given by Eq. (2.21):

$$Q_i = n\{1 - n / N)\}b_o \, f(\sigma) \, g(w) \, \partial u_i^P / \partial z \tag{2.21}$$

where f and g are material functions to be determined by experiment.

The Stress Distribution in a Granular Domain.

Attention must be given to this aspect of constitutive behavior of a granular medium. There are in fact many subtle points which are not immediately evident upon initial examination. The essence of the problem lies in the fact that the stress is transferred from the mobile phase to the elastic skeleton by friction. It is the friction that inhibits the collapse of a granular aggregate when the latter is subjected to surface tractions.

The simplest possible model that bears resemblance to the physics of the motion is that of a friction block (the mobile phase) moving between two elastic walls, which represent the elastic skeleton. See Figure 1. The block is of much larger cross-sectional area than that of the walls, since the mobile particles are much more numerous than their stationary counterparts. The depth of the block reflects the extent to which the stress has been transmitted to the interior of the mobile phase.

One can see that, at the surface, most of the stress is carried by the block, (the mobile phase). However, in the interior, and in a manner consistent with experimental data, the skeleton carries a larger stress commensurate

with the depth, as stress is transferred from the block to the elastic skeleton (the walls) by friction.

Thus letting T_i^e denote the surface traction on the elastic phase, T_i^p the traction on the mobile phase, and T_i the traction on the surface of the granular medium, then by simple proportion:

$$T_i^e = \{1 - (n / N)\} T_i \quad ; \quad T_i^p = (n / N) T_i \tag{2.22a,b}$$

Hence:

$$T_i = T_i^e + T_i^p \tag{2.23}$$

Some further observations are in order. It follows from Eq. (2.14) that:

$$\partial \psi / \partial u_i^p{}_{,j} = \sigma_{ij}^p \tag{2.24}$$

where σ_{ij}^p is the stress in the mobile phase. Thus Eq. (2.15) reads:

$$\sigma_{ij}^p{}_{,j} = Q_i \tag{2.25}$$

Thus Qi acts as a negative body force brought about by the interaction between the mobile and elastic phase. Since action and reaction are equal and opposite, an equal and opposite force must be exerted on the elastic phase. It follows that the equilibrium equation for the elastic phase must read:

$$\sigma_{ij}^e{}_{,j} + Q_i = 0 \tag{2.26}$$

Furthermore as a result of Eqs. (2.25) and (2.26):

$$\sigma_{ij}^e + \sigma_{ij}^p = \sigma_{ij} \tag{2.27}$$

$$\sigma_{ij},_j = 0 \tag{2.28}$$

where σ_{ij} is the resultant stress in the granular medium.

THERMODYNAMICS OF THE ELASTIC PHASE

We conclude this subsection with the treatment of the thermodynamics of the elastic phase. Grain mobility and deformability are independent processes in the sense that they pertain to disjoint subsets of grains. It is reasonable, therefore, to expect the free energy density ψ of the aggregate domain to be the sum of the individual densities of the elastic and mobile phase. Thus:

$$\psi = \psi_p + \psi_e \tag{2.29}$$

where

$$\psi_p = \psi_p\left(\frac{\partial u_i^p}{\partial x_j}\right) \tag{2.30}$$

while

$$\psi_e = \psi_e\left(\frac{\partial u_i^e}{\partial x_j}\right) \tag{2.31}$$

The essential difference between the elastic and the mobile phase is that in the elastic phase, the dissipation is zero, while a body force field Q_i arises as a result of the dissipation in the plastic phase.

Thus in the case of the elastic phase the variational equation (2.10a) will read

$$\int_S T_i^e \delta u_i^e + \int_D Q_i \; \delta u_i^e = \delta \Psi_e \tag{2.32}$$

since in fact $f_i = Q_i$. Following the technique previously applied to the plastic phase we find that

$$T_i^e = \left(\frac{\partial \psi}{\partial u_{i,j}^e} \right) n_j \qquad \text{on S} \tag{2.33}$$

$$\left(\frac{\partial \psi}{\partial u_{i,j}^e} \right)_{,j} + Q_i = 0 \qquad \text{in D} \tag{2.34}$$

of course we recognize $\partial\psi/\partial u_{i\,j}^e$ as σ_{ij}^e so that Eq. (2.34) is in conformance with Eq. (2.26) which was obtained by static considerations.

The Displacement Field

Following the previous notation u_i^e is elastic displacement field pertaining to the skeleton, where u_i^p is the displacement field of the mobile phase. Let u_i be the displacement field of the aggregate. Evidently, at the surface, the three fields must satisfy the condition

$$\int_S T_i^e \delta u_i^e \; ds + \int_S T_i^p \; \delta u_i^p \; dS = \int_S T^i \; \delta u_i \; dS \tag{2.35}$$

at this point we utilize Eqs. (2.22a,b) to obtain

$$(1 - n / N)\delta u_i^e + \frac{n}{N} \delta u_i^p = \delta u_i \tag{2.36}$$

which is identical to Eq. (2.1a). Equation (2.36) relates the displacement field of the aggregate to the displacements of the individual phases.

It is observed that the skeleton is not static. It changes configuration and is not always made up of the same particles. Nonetheless on physical grounds the compliance of the skeleton is directly related to the deformability of the individual particles. The deformation of the skeleton is small compared to the aggregate deformation due to particle motion. Therefore we expect that

$$\left\| \delta u_i^e \right\| << \left\| \delta u_i^p \right\| \tag{2.37}$$

and since, furthermore, the number of mobile particles n is much larger than that of the stationary counterparts (as estimated values of n/N is 0.8), one may ignore the term $(1 - n/N)u_i^e$ relative to $n/N\ u_i^p$ in Eqs. (2.36). Hence, to a good degree of accuracy

$$\delta u_i = \frac{n}{N} \delta u_i^p \tag{2.38}$$

<u>Equation (2.18) in One Dimension</u>

In one dimension ($u_1^p = u^p$, $u_2^p = u_3^p = 0$) Eq. (2.18) becomes:

$$\frac{\partial \sigma^p}{\partial x} = b \frac{\partial u^p}{\partial z} \tag{2.39}$$

where

$$\sigma^p = \frac{\partial \psi}{\partial \left(\partial u^p / \partial x \right)} \tag{2.40}$$

Within the restriction of small strains

$$\psi = \frac{1}{2} A \left(\frac{\partial u}{\partial x} \right)^2 \tag{2.41}$$

where A is an appropriate elastic stiffness coefficient. (In general A will depend on the confining pressure as well as on the porosity and, therefore, on the plastic volumetric strain $(\partial u^p/\partial x)$). It follows from Eqs. (2.39) and (2.40) that:

$$\sigma^p = A\frac{\partial u^p}{\partial x} + \frac{1}{2}\frac{\partial A}{\partial\left(\partial u^p / \partial x\right)}\left(\frac{\partial u}{\partial x}\right)^2 \tag{2.42}$$

Since in the linear theory terms in $(\partial u^p/\partial x)^2$ are ignorable, then

$$\sigma^p = A\frac{\partial u^p}{\partial x} \tag{2.43}$$

To make matters simple, the dependence of A on σ and w is assumed to be of the same as that of b. Thus:

$$A = A_o f(\sigma) g(w) \tag{2.44}$$

and hence:

$$\sigma^p = A_o f(\sigma) g(w)\frac{\partial u^p}{\partial x} \tag{2.45}$$

At this point, we substitute Eq. (2.45) in Eq. (2.39), retain linear terms and use Eq. (2.21) to find:

$$A_o\frac{\partial^2 u^p}{\partial x^2} = N(1 - n/N) b_o \frac{\partial u^p}{\partial x} \tag{2.46}$$

This is the equation that we shall use in the determination of the internal displacement field in simple compression generated experimentally by Gill [4,6].

Explicit Three-Dimensional Equations

In the context of small strains and an initially isotropic granular medium we set

$$\psi_P = \frac{1}{2}\lambda\left(\varepsilon^P_{ii}\right)^2 + \mu\, \varepsilon^P_{ij}\, \varepsilon^P_{ij} \tag{2.47}$$

where the elastic constants λ and μ will depend on the porosity as well as the prevailing hydrostatic stress. We now substitute Eq. (2.47) in Eq. (2.48), perform the indicated differentiation and retain only the linear terms in the displacement gradient to obtain the following equations for the plastic displacement field u^P_i

$$(\lambda + \mu)u^P_{j,ji} + \mu\, u^P_{i,jj} = b\frac{\partial u^P_i}{\partial z} \tag{2.48}$$

We now express u^P_i in terms of its (independent) irrotational, and isotropic components $\bar{u}_i$ and u^*_i, i.e.,

$$u^P_i = \bar{u}_i + u^*_i \tag{2.49}$$

such that

$$u^*_{i,j} = 0 \quad ; \quad e_{ijk}\, u_{k,j} = 0 \tag{2.50a,b}$$

where e_{ijk} is the permutation symbol. This decomposition allows Eq. (2.48) to be expressed into two separate relations. Using Eq. (2.50a,b) Eq. (2.48) becomes:

$$(\lambda + \mu)\bar{u}_{i,ji} + \mu\, \bar{u}_{i,jj} + \mu\, u^*_{i,jj} = b\frac{\partial \bar{u}_i}{\partial z} + b\frac{\partial u^*_i}{\partial z} \tag{2.51}$$

in view of Eq. (2.50a). However, in view of Eq. (2.50b)

$$\bar{u}_{i,j} = \bar{u}_{j,i} \tag{2.52}$$

Thus, Eq. (2.51) becomes

$$(\lambda + 2\mu)\bar{u}_{j,ji} + \mu\, u^*_{i,jj} = b\frac{\partial \bar{u}_i}{\partial z} + b\frac{\partial u^*_i}{\partial z} \tag{2.53}$$

Since $\bar{u}_i$ and u^*_i are independent fields:

$$\mu\, u^*_{i,jj} = b\frac{\partial u^*_i}{\partial z} \tag{2.54}$$

and

$$(\lambda + 2\mu)\bar{u}_{j,ji} = b\frac{\partial \bar{u}_i}{\partial z} \tag{2.55}$$

Now, because $\bar{u}_i$ is irrotational, there exists a potential field such that

$$\bar{u}_i = \frac{\partial \phi}{\partial x_i} \quad . \tag{2.56}$$

Hence $\bar{u}_i$ will satisfy Eq. (2.55) if

$$(\lambda + \mu)\nabla^2 \phi = b\frac{\partial \phi}{\partial z} \tag{2.57}$$

Thus the motion is decomposed into a deviator field and a volumetric field. The deviator field obeys Eq. (2.54) while the isotropic field obeys Eq. (2.57).

Probabilistic Considerations

In support of the thermodynamic arguments discussed previously we pursue a probabilistic theory akin to the one that governs diffusion in liquids with Brownian motion [11]. To this end let $f(x_i)$ be a property of the medium

where x_i are the coordinates of the particles of the medium. Let a disturbance cause a change in x_i which we denote by Δx_i, in time Δt. We note that microscopic changes are discreet and that Δt is the average minimum time over which a microscopic change can take place. Thus,

$$\Delta x_i = v_i \, \Delta t \tag{2.58}$$

where $\Delta t \equiv \tau$ is a material property of the process, a "characteristic time," so to speak, and v_i is the particle velocity vector. The field Δx_i is not deterministic, (i.e., it is random) in the sense that the probability of a velocity v_i is p, where

$$p = \phi(v_i) \tag{2.59}$$

We now assume that motion in three mutually perpendicular directions constitutes three independent events, in which case,

$$\phi(v_i) = \phi(v_1)\phi(v_2)\phi(v_3) \tag{2.60}$$

and let

$$\int_{-\infty}^{\infty} \phi(v)dv = 1 \quad , \quad \int_{-\infty}^{\infty} v \, \phi(v)dv = \bar{v} \quad , \quad \int_{-\infty}^{\infty} v^2\phi(v)dv = \overline{v^2} \tag{2.61a,b,c}$$

In the absence of externally applied <u>body</u> forces, we stipulate that $\phi(v)$ is a symmetric function in the sense that forward and backward motion are equally likely. Thus,

$$\bar{v} = 0 \tag{2.62}$$

We note that $\overline{v^2}$ is a property of the distribution $\phi(v)$.

Following a disturbance, the expected value $\bar{f}$ of f is:

$$\bar{f} = \iiint_{-\infty}^{\infty} f(x_i + v_i \Delta t)\phi(v_i) dv_i \tag{2.63}$$

Note that the expected value $\bar{f}_o$ in the undisturbed configuration is

$$\bar{f}_o = \iiint_{-\infty}^{\infty} f(x_i)\phi(v_i) dv_i = f(x_i) \tag{2.64}$$

in view of Eq (2.61a). Thus,

$$\Delta \bar{f} = \bar{f} - \bar{f}_o = \iiint_{-\infty}^{\infty} \{f(x_i + v_i \Delta t) - f(x_i)\}\phi(v_i) dv_i \tag{2.65}$$

Limiting the study to disturbances that are of small order, say O(ε):

$$f(x_i + \Delta u_i) - f(x_i) = f_{,i} \Delta u_i + \frac{1}{2} f_{,ij} \Delta u_i \Delta u_j + 0(\varepsilon^3) \tag{2.66}$$

Thus,

$$\Delta\bar{f} =$$

$$\Delta t \int\int\int_{-\infty}^{\infty} \left(\frac{\partial f}{\partial x_1} v_1 + \frac{\partial f}{\partial x_2} v2 + \frac{\partial f}{\partial x_3} v_3 \right) \phi(v_1)\phi(v_2)\phi(v_3) dv_1 dv_2 dv_3$$

$$+ \frac{\Delta t^2}{2!} \int\int\int_{-\infty}^{\infty} \left(\frac{\partial^2 f}{\partial x_1^2} v_1^2 + \frac{\partial^2 f}{\partial x_2^2} v_2^2 + \frac{\partial f}{\partial x_3^2} v_3^2 \right) \phi(v_1)\phi(v_2)\phi(v_3) dv_1 dv_2 dv_3$$

$$+ \frac{2}{2!} \Delta t^2 \int\int\int_{-\infty}^{\infty} \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} v_1 v_2 + \frac{\partial^2 f}{\partial x_1 \partial x_3} v_1 v_3 + \frac{\partial^2 f}{\partial x_2 \partial x_3} v_2 v_3 \right) \phi(v_1)\phi(v_2)\phi(v_3) dv_1 dv_2 dv_3$$

(2.67)

which in view of Eq. (2.72), becomes,

$$\Delta\bar{f} = \frac{\tau^2}{2!} \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \right) \overline{v^2} \tag{2.68}$$

But

$$\Delta\bar{f} = \frac{\partial f}{\partial t} \Delta t = \frac{\partial f}{\partial t} \tau \tag{2.69}$$

and hence:

$$\frac{\partial \bar{f}}{\partial t} = D \nabla^2 \bar{f} \tag{2.70}$$

where ∇^2 denotes the Laplacian operator and

$$D = \frac{\overline{v^2}\tau}{2} \tag{2.71}$$

i.e., the diffusion coefficient.

However, τ is a characteristic material time. Thus we may define an intrinsic time z, such that

$$dz = \frac{dt}{\tau} \tag{2.72}$$

in which case Eq. (2.70) becomes:

$$\frac{\partial \bar{f}}{\partial z} = a\ \tau^2 \bar{f} \tag{2.73}$$

where

$$a = \frac{1}{2}\overline{v^2} \tag{2.74}$$

Specifically in the case where $\bar{f}$ is the isochronic displacement fields u_i^*, then

$$\frac{\partial u_i^*}{\partial z} = a^* \nabla^2 u_i^* \tag{2.75}$$

where

$$a^* = \frac{k}{b} \tag{2.76}$$

while if $\bar{f}$ is the irrotational displacement field potential ϕ, then

$$\frac{\partial \phi}{\partial z} = \bar{a}\ \nabla^2\ \phi \tag{2.77}$$

where

$$\bar{a} = \frac{\lambda + 2k}{b} \tag{2.78}$$

Thus probability theory and irreversible thermodynamics have been brought into complete correspondence.

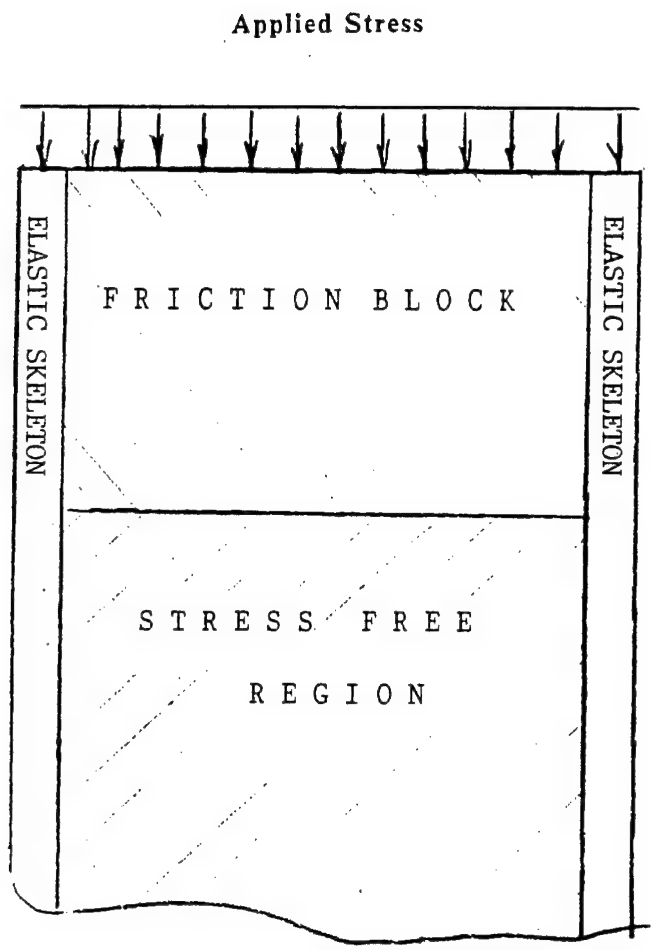


Fig. 1. Friction Block Model

SECTION 3

APPLICATIONS

DISPLACEMENT FIELD UNDER CONSTRAINED AXIAL COMPRESSION

Case (1) Monotonic Loading.

The theory has been applied to the case of vertical motion of particles in a cuboid under conditions of axial strain. Insofar as experiment does not predict uniform strain under uniform compressive stress, the theoretical consequences were put to the test by solving, with a finite difference technique the diffusion equation derived from Eq. (18), for the particle displacement at the midway plane of the cuboid. Excellent agreement was obtained between theory and measurement. This outcome is encouraging in that it supports the conceptual framework of the theory which is essential to the understanding of the deformation of large bulks of granular domains.

The same technique was also used to obtain theoretically the particle displacement at points located at one quarter and three-quarters of the height of the cuboid. Experimental results at these points were not available for comparison. An analytical solution was also obtained by an "infinite length" approximation. This solution is in closed form and also agrees with experiment.

Finite Difference Scheme.

We proceed to solve by a finite difference technique the one-dimensional generic diffusion equation

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial z} \tag{3.1}$$

where

$$a = (A_o/b_o)(1 - n/N)^{-1}N^{-1} \tag{3.1a}$$

Let the height h of the cuboid be divided into N+1 equal intervals, with a resulting number of N interior nodes and two boundary nodes one at each boundary.

The differential Eq. (3.1) is then satisfied at all interior points, thus giving rise to a set of simultaneous ordinary differential equations shown below:

$$(2u_1 - u_2) + \frac{1}{C_N}\frac{du_1}{dz} = 0$$

$$(-u_1 + 2u_2 - u_3) + \frac{1}{C_N}\frac{du_2}{dz} = 0 \tag{3.2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(-u_{N-1} + 2u_N - U) + \frac{1}{C_N}\frac{du_N}{dz} = 0$$

where

$$C_N = \frac{a}{[n/(N+1)]^2} \tag{3.3}$$

Equations (3.2) may be written in the matrix form

$$[A]\{u\} + \frac{1}{C_N}\left\{\frac{\partial u}{\partial z}\right\} = \{U\} \tag{3.4}$$

where

$$[A] = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & \cdots & \cdots & \cdots & \\ & & & -1 & 2 \end{bmatrix} \tag{3.5}$$

and

$$U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ U \end{pmatrix} \tag{3.6}$$

There exists a diagonalizing matrix R_{ij} where

$$R_{ij} = \frac{1}{\left(\frac{N+1}{2}\right)^{1/2}} \sin\left(\frac{ij\,\pi}{N+1}\right) \tag{3.7}$$

such that

$$R^T A\, R = [\lambda] \tag{3.8}$$

$$\lambda_n = 4 \sin^2 \frac{n\,\pi}{2(N+1)} \tag{3.9}$$

Define:

$$u^* = R_u^T \quad , \quad u = R_u^* \tag{3.10a,b}$$

$$C_N\left\{R^T A\, R\right\}u^* + \frac{du^*}{dz} = R^T U = U^* \tag{3.11}$$

Then, in view of Eqs. (3.8) and (3.9) the following set of equations result:

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{pmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{pmatrix} + \frac{1}{c_N} \begin{pmatrix} \frac{du_1^*}{dz} \\ \vdots \\ \frac{du_n}{dz} \end{pmatrix} = \begin{pmatrix} U_1^* \\ \vdots \\ U_n^* \end{pmatrix} \tag{3.12}$$

which are N uncoupled ordinary differential equations of the form

$$\lambda_r\, u_r^* + \frac{1}{C_N}\frac{du_r^*}{dz} = U_r^*(z) \tag{3.13}$$

Knowing U_r^* one may then solve Eq. (3.13) and use Eq. (3.10b) to determine u_r as a function of z all the modes r = 1 ... N.

To calculate theoretically the mean vertical displacement of particles located points h/4, h/2 and 3h/4 from the base of the cuboid, we subdivided the height h into four equal intervals with a resulting number of <u>three</u> interior nodes. Thus in Eq. (3.3), N = 3.

<u>Case of N = 3</u>

Use of Eqs. (3.7) and (3.8) gives rise to the following results:

$$R_{ij} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \tag{3.14}$$

$$[\lambda] = \begin{bmatrix} 2\left(1-\frac{\sqrt{2}}{s}\right) & & \\ & 2 & \\ & & 2\left(1+\frac{\sqrt{2}}{2}\right) \end{bmatrix} \tag{3.15}$$

Also,

$$C_3 = \frac{1}{(h/4)^2} \tag{3.16}$$

Solution of Eq. (3.13)

It follows from Eqs. (3.6), 3.10b) and 3.14) that

$$U_r^* = b_r\, U \tag{3.17}$$

where

$$b_r = \left\{ \frac{1}{2} \quad -\frac{\sqrt{2}}{2} \quad \frac{1}{2} \right\} \tag{3.18}$$

In view of our discussion in Section 2, under monotonic conditions,

$$dz = d\left(\frac{\partial u}{\partial x}\right) \tag{3.19}$$

In keeping with the linear form of the theory we take z to be its average value $\bar{z}$ over the entire domain of the cuboid. Thus

$$dz = d\bar{z} = \frac{1}{h}\int_{o}^{h} d\left(\frac{\partial u}{\partial x}\right)dx = \frac{1}{h}d\int_{o}^{h}\left(\frac{\partial u}{\partial x}\right)dx = \frac{dU}{h} \tag{3.20}$$

and hence:

$$dz = \frac{dU}{h} \tag{3.21}$$

at all points along the height of the cuboid.

In view of Eq. (3.21), Eq. (3.13) becomes:;

$$\lambda_r u_r^* + \frac{h}{c^*}\frac{du_r^*}{dU} = b_r U \tag{3.22}$$

This has been solved by Laplace transform in the light of the initial condition $u_r^*(0) = 0$, to give

$$u_r^* = \frac{b_r}{\lambda_r}\left\{U - \frac{b_r}{\lambda_r}\frac{1}{c^*\lambda_r}\left(1 - e^{-c^*\lambda_r U}\right)\right\} \tag{3.23}$$

where

$$c^* = \frac{C_3}{h} \tag{3.24}$$

We remark that the only material constant in the above equation is c^*.

Evidently u_r may be obtained from u_r^* by the relation

$$u_r = R_{rs}\, u_s^* \tag{3.25}$$

following Eq. (3.10b) and the fact that $R = R^T$.

Since Eq. (3.1) is a <u>generic</u> equation we obtain the solution to our problem by setting

$$u_r^P \equiv u_r \tag{3.26}$$

<u>Computations</u>

It is clear from Eq. (3.33) that the only material constant is c^*. To obtain c^* we made use of experimental measurements made by Gill [4] at a "point" close to the mid-height of the cuboid, and derived the theoretical expression for N = 1. Omitting the analysis which was given previously [2],

$$u^p = \frac{1}{2}\left[U - \frac{1}{c}\left(1 - e^{-cU}\right)\right] \tag{3.27}$$

where U is the measured displacement and c = 1/2 c*. Specifically

$$c = 8\frac{A_o}{b_o}\left(\frac{1}{N-n}\right)\frac{1}{h^3} \tag{3.28}$$

Good agreement with measured values of u^p was found when c was set equal to 0.6, i.e., c* = 1.2.

When u^p was calculated at h/2 for this value of c* using N = 3, the agreement with observation, though good, was not quite as satisfactory. For this reason the theoretical prediction was probed for values of c* = 0.9 and 1.0 as well.

The results are shown in the following Tables:

c* = 0.9

u^p, U in mm.

	U = 1	U = 2	U = 3
u_1^p	0.010	0.085	0.219
u_2^p	0.063	0.293	0.635
u_3^p	0.280	0.821	1.463

c* = 1.0

	U = 1	U = 2	U = 3
u_1^p	0.013	0.104	0.251
u_2^p	0.073	0.316	0.677
u_3^p	0.298	0.863	1.519

$c^* = 1.2$

	U = 1	U = 2	U = 3
u_1^p	0.02	0.142	0.315
u_2^p	0.094	0.361	0.760
u_3^p	0.334	0.947	1.630

Notation: $u_1^p = u^p|_{h/4}$, $u_2^p = u^p|_{h/2}$, $u_3^p = u^p|_{3h/4}$.

A plot of u^p versus x for various values of U is shown in Figure 2, while plots of u^p versus U at different points along the height of the cuboid are shown in Figure 3. Experimental points are also shown at x = 0.56 h.

An Exact Solution

In Figure 3, the decay of u^p with respect to x is sufficiently fast to warrant the question of whether the fixed end of the cuboid is not sufficiently far from the free end to warrant the treatment of the length of the cuboid as "infinite". We show in this sub-section that this is in fact the case. However, dz is still set equal to dU/h where h is a characteristic length, the actual length of the cuboid. The value of h is relevant only insofar as it concerns the correlation of c* to the value of the diffusion constant a/h in the generic equation (3.1) which now becomes:

$$\frac{a}{h}\frac{\partial^2 u^p}{\partial x^2} = \frac{\partial u^p}{\partial U} \tag{3.29}$$

To proceed with the solution of this equation we take its Laplace transform in light of the initial condition $u^p(o) = 0$. Thus:

$$\left(\frac{a}{h}\right)\frac{\partial^2 \bar{u}^p}{\partial x^2} = p\,\bar{u}^p \tag{3.30}$$

where p is the Laplace transform parameter.

If c* the length of the cuboid is infinite, then the boundedness condition on u as $x \rightarrow \infty$ reduces the solution of Eq. (3.30) to the form

$$\bar{u}^{p} = Ae^{-\left(x/\sqrt{\frac{a}{h}}\right)p^{1/2}} \tag{3.31}$$

Now at x = 0, u = U and thus $\bar{u} = 1/p^2$. Hence,

$$\bar{u}^{p} = \frac{1}{p^{2}}e^{-\left(x/\sqrt{\frac{a}{h}}\right)p^{1/2}} \tag{3.32}$$

However, we note that

$$L^{-1}\left\{\frac{1}{p}e^{-\left(x/\sqrt{\frac{a}{h}}\right)p^{1/2}}\right. = 1 - \operatorname{erf}\frac{x}{2\sqrt{\frac{a}{h}}\sqrt{U}} \tag{3.33}$$

where L^{-1} is the inverse Laplace Transform operator and erfx is the Error function. Thus:

$$u^{p} = \int_{o}^{U}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{\frac{a}{h}}}\frac{1}{\sqrt{U'}}\right)dU' \tag{3.34}$$

which is a closed form (exact) solution for u^p.

Alternatively,

$$\frac{du^{p}}{dU} = 1 - \operatorname{erf}\left(\frac{(x/h)}{2\sqrt{\frac{a}{h^{3}}}} \bullet \frac{1}{\sqrt{U}}\right) \tag{3.25}$$

Letting

$$ {}^{x}\!/_{h} \equiv y \quad \text{and} \quad k = \frac{1}{\sqrt[2]{\frac{1}{h^3}}} \ , $$

$$ \frac{du^p}{dU} = 1 - \operatorname{erf}\left\{ k \frac{y}{\sqrt{U}} \right\} \tag{3.26} $$

Now it follows from Eq. (3.28) that

$$ c^* = 16\left(\frac{a}{h^3} \right) = 1 \tag{3.27} $$

Hence, $\sqrt{\frac{a}{h^3}} = \frac{1}{2}$ and $k = 2.$

Thus,

$$ \frac{du^p}{dU} = 1 - \operatorname{erf}\left(\frac{2y}{\sqrt{U}} \right) \tag{3.28} $$

Below is a table of values for du^p/dU for both the finite difference solution when N = 3 and the exact solution given by Eq. (3.27) with the asymptotic assumption of infinite length. The comparison is given at the two stations x = h/4 and x = h/2. In the case of x = h/2 a comparison with the experimental values is also given. The agreement is close

<u>x = h/4</u>

U	1	2	3
u'(N=3)	0.47	0.61	0.69
u'(Exact)	0.45	0.61	0.72

$x = h/2$

U	1	2	3
u'(N=3)	0.132	0.32	0.42
u'(Exact)	0.134	0.32	0.42
u' (Exp.)	0.16	0.33	0.42

$$u' \equiv \frac{du^p}{dU}$$

Case (2). Unloading-

One set of experiments by Gill [4] probed the axial displacement field under conditions of unloading. The result of one typical such experiment is shown in Figure 4. The essential finding is that, upon unloading the relative u^p versus U is essentially a straight line parallel to the line oB. We observe that the line oB is the response appropriate to a local theory ("local behavior") which in the context of the present theory can only be an elastic response in the light of Eq. (3.39).

The inference is drawn as follows. During elastic deformation $du^p/dz = 0$, which implies (in view of Eq. (2.1)) that,

$$n = 0 \tag{3.28}$$

Hence, during unloading, all particles are immobile and the deformation of the aggregate results from the elastic expansion of the grains themselves.

Thus during unloading there is no mobile phase. The entire aggregate domain is an elastic phase and hence $Q_i = 0$ in Eq. (2.34). One concludes that during unloading (where u_i^e and ψ are measured relative to the reference state at the onset of unloading)

$$\sigma^e_{ij},_j = \left(\frac{\partial \psi}{\partial u^e_i,_j} \right)_{,j} = 0 \quad \text{in D} \tag{3.29}$$

In the one-dimensional case,

$$\sigma^e = \left(\frac{\partial \psi}{\partial u_x}\right) = \text{constant} = \sigma \tag{3.30}$$

and hence

$$\frac{\partial u^e}{\partial x} = \frac{\partial u}{\partial x} = \quad \text{constant} \tag{3.31}$$

where u is measured relative to the reference state at the onset of unloading. Thus the theory is local and u^e is a linear function of x, a fact that agrees with the experimental data of Gill. See Figure 5.

Solution of Eq. (3.31) yields the result:

$$u = U\left(\frac{x}{h}\right) \tag{3.31a}$$

since u = U at x = h.

The Stress Response Under Uniaxial Compression

Loading Case

We begin with Eqs. (2.45) and (2.22b) to find the following relation for σ:

$$\sigma = \frac{N}{n} A_o \, f(\sigma) g(w) \frac{\partial u^P}{\partial x} \tag{3.32}$$

The constants (N/n) and A_o in conjunction with the function f define a new function $F(\sigma) = (N/n)A_o f(\sigma)$, so that σ may be written in the form

$$\sigma = F(\sigma)g(w)\frac{\partial u^P}{\partial x} \tag{3.32a}$$

We point out that because the theory is non-local w varies pointwise along the height of the cuboid. However, because of the small strain involved, the porosity does not vary substantially during deformation. Therefore the dependence of g on w is weak. For this reason and to simplify the analysis in this investigative stage of the work, we regard w as the average porosity of the cuboid.

To express the effect of w analytically we reason that the porosity depends on the hydrostatic strain. This is true in principle even though we realize that porosity, strictly, is due to positional arrangement of the grains. Thus we set

$$w = w(\epsilon) \tag{3.32b}$$

and

$$g[N(\epsilon)] = G(\epsilon) \tag{3.32c}$$

Thus, Eq. (3.32) becomes,

$$\sigma = F(\sigma)G(\epsilon)\frac{\partial u^P}{\partial x}$$

This is an implicit relation between σ and $\partial u^P / \partial x$.

<u>Remark</u>. There exists a philosophical question of determination insofar as the function $G(\epsilon)$ is concerned. The difficulty lies in the fact that the strain in the reference state is not known and is conveniently assumed to be zero. However, following a history of loading and unloading to zero stress, there remains a residual strain which requires a change in porosity. Thus during a reloading history from that point, ϵ cannot be set equal to zero, but to ϵ_0 which denotes the residual strain. If such a history were not known then ϵ_0 would be an unknown constant. Hence $G(\epsilon)$ is in fact ,

$$G = G(\epsilon_0 + \epsilon) \; . \tag{3.33}$$

where ϵ_0 is a function of previous history and is not known. The problem is obviated if G is such a function that:

$$G(\epsilon_0 + \epsilon) = k\, G(\epsilon) \tag{3.34}$$

where $k = k(\epsilon_0)$ and, therefore, a constant. Equation (3.34) is satisfied if

$$G(\epsilon) = e^{\beta\epsilon} \tag{3.35}$$

where β is a material constant.

Thus Eq. (3.32) becomes

$$\sigma = F(\sigma) e^{\beta\epsilon} \frac{\partial u^P}{\partial x} \tag{3.36}$$

Since the constant k in Eq. (3.34) can be absorbed in the function F, one can set without difficulty $\epsilon_0 = 0$ in the reference state, i.e., prior to the commencement of a strain history. We shall see that experiments by Gill [6] do in fact corroborate the form of G depicted in Eq. (3.35).

The Stress at the Moving Boundary

The experimental results by Gill [6] were presented by plotting the stress σ versus the displacement U at the moving boundary, but also versus the "strain" U/h. Of course the two presentations are equivalent.

A plot of the experimentally obtained axial stress plotted versus axial strain is shown in Figure 5, for a history of three loading paths oa_1, a_2a_3, a_4a_5 and three unloading paths a_1a_2, a_3a_4, a_5a_6. In our discussion of the experimentally obtained mean particle displacement in the neighborhood close to the midpoint of the cuboid we reasoned that unloading behavior is elastic.

This conclusion implies that the unloading paths are representative of elastic behavior. If that is the case then the curves a_2a_1, a_4a_3 and a_6a_5 should all be part of one and the same curve. This is, in fact, the case. The elastic response is represented quite accurately by the analytical form

$$\sigma = 0.21\left(\epsilon^{e}\right)^{6.25} \tag{3.37}$$

as shown in Figure 6. The analytical representation is not of central importance. Of importance is the fact that all three unloading stress-strain curves are part of one and the same curve.

We observe that, in a manner consistent with our proposed physics of granular motion, loading (reloading) is not an elastic process because the mobility is not zero. Thus the curve a_0a_1 does not coincide with the unloading curve a_1a_2 and neither do the other corresponding curves. Clearly one cannot depict such results by means of a yield surface!

To determine the stress during loading at the moving boundary of the cuboid, we make use of the "infinite length" idealization, which as we have seen, gave an accurate displacement distribution in that it matched the finite difference solution obtained using he actual length of the cuboid.

To this end we begin with Eq. (3.32) and differentiate both sides with respect to x to find

$$\frac{d\bar{u}^{p}}{dx} = \frac{1}{\sqrt{\frac{a}{h}}^{\,3/2}}\, e^{-\left(2\Big/\sqrt{\frac{a}{h}}\right)p^{1/2}} \tag{3.38}$$

Specifically at x = 0, i.e., the moving boundary,

$$\frac{d\bar{u}^P}{dx} = \frac{1}{\sqrt{\frac{a}{h}}} p^{-3/2} \tag{3.39}$$

Upon taking inverse transform of Eq. (3.39) we find:

$$\frac{du^P}{dx} = \frac{1}{\sqrt{\frac{a}{h}}} \sqrt{\pi U} \tag{3.40}$$

or

$$\frac{du^P}{dx} = \sqrt{\frac{\pi h^2}{a}} (\epsilon)^{1/2} \tag{3.41}$$

where $\epsilon = U/h$.

At this point we utilize Eq. (3.32) and set

$$F^* = \sqrt{\frac{\pi h^2}{a}} F \tag{3.42}$$

to obtain the following expression for the stress at the free boundary:

$$\sigma = F^*(\sigma) e^{\beta U/h} \epsilon^{1/2} \tag{3.44}$$

Equation (3.44) is the basic equation that will be used for the theoretical depiction of the loading curve $a_o a_1$, i.e., the <u>initial</u> loading curve.

<u>Discussion</u>. Every point on the stress-strain path in Figure 6 must lie on a solution curve of the partial differential equation (3.29). The unloading paths satisfy Eq. (3.29) identically since on those paths $u^P = 0$. The loading paths must satisfy Eq. (3.29) but with different initial conditions.

We observe, however, that Eq. (3.29) remains invariant with the "translation transformation".

$$U' = U - U_o \tag{3.45a}$$

$$u^{p'} = u^p - u_o^p \tag{3.45b}$$

Thus Eq. (3.41), i.e., the solution for path oa_1 applies to path a_2a_3, if U_o and u_o^p are the values of these variables at the point a_2. Hence,

$$\frac{\partial u^{p'}}{\partial x} = \sqrt{\frac{\pi h^2}{a}} (\epsilon')^{1/2} \tag{3.46}$$

However, insofar as the function G in Eq. (3.32) is concerned, ϵ and, therefore, U are related to the actual total change in the mean porosity and therefore must be measured relative to the initial reference point 0. Thus for the loading paths a_2a_3 and a_4a_5, Eq. (3.44) will read

$$\sigma = F^*(\sigma) e^{\beta\epsilon} (\epsilon')^{1/2} \tag{3.47}$$

there ϵ is the <u>cumulative</u> strain (i.e., the strain measured from the reference point 0) and ϵ' is measured from the point of reloading.

The comparison between theory and experiment is shown in Figure 7 with $\beta = 0.06$ and $F^*(\sigma)$ as shown in Figure 8. How β and F^* are determined will be the subject of the next subsection.

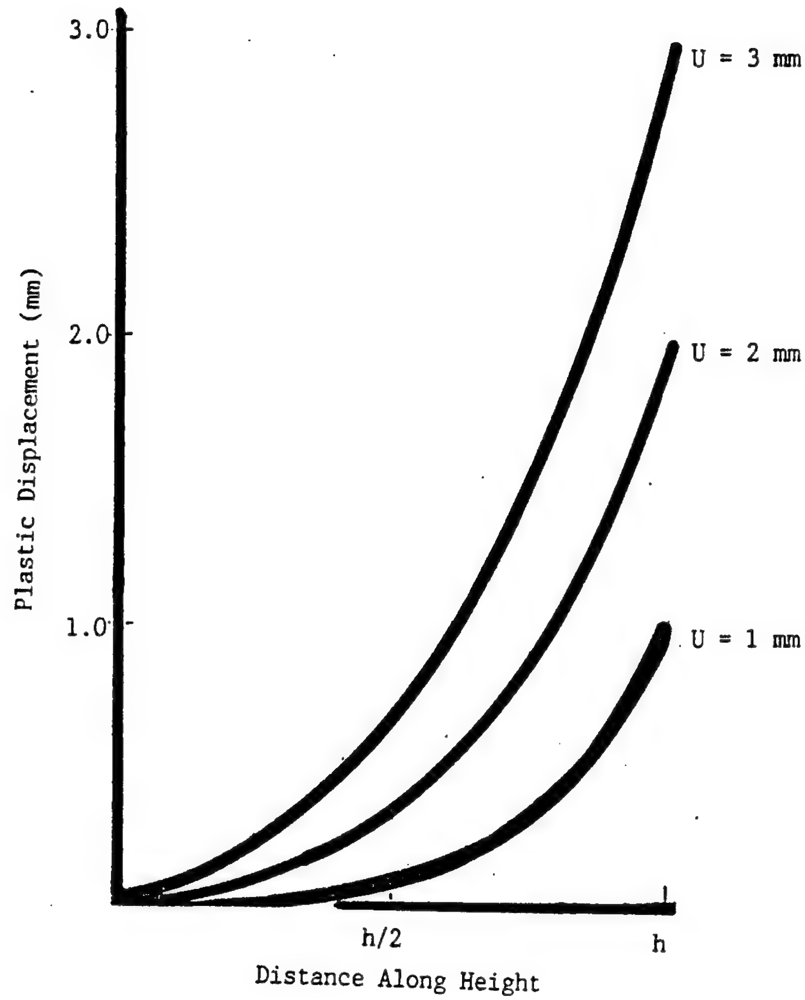


Fig. 2. Theoretical Plastic Displacement vs Height for Various Values of Surface Displacement

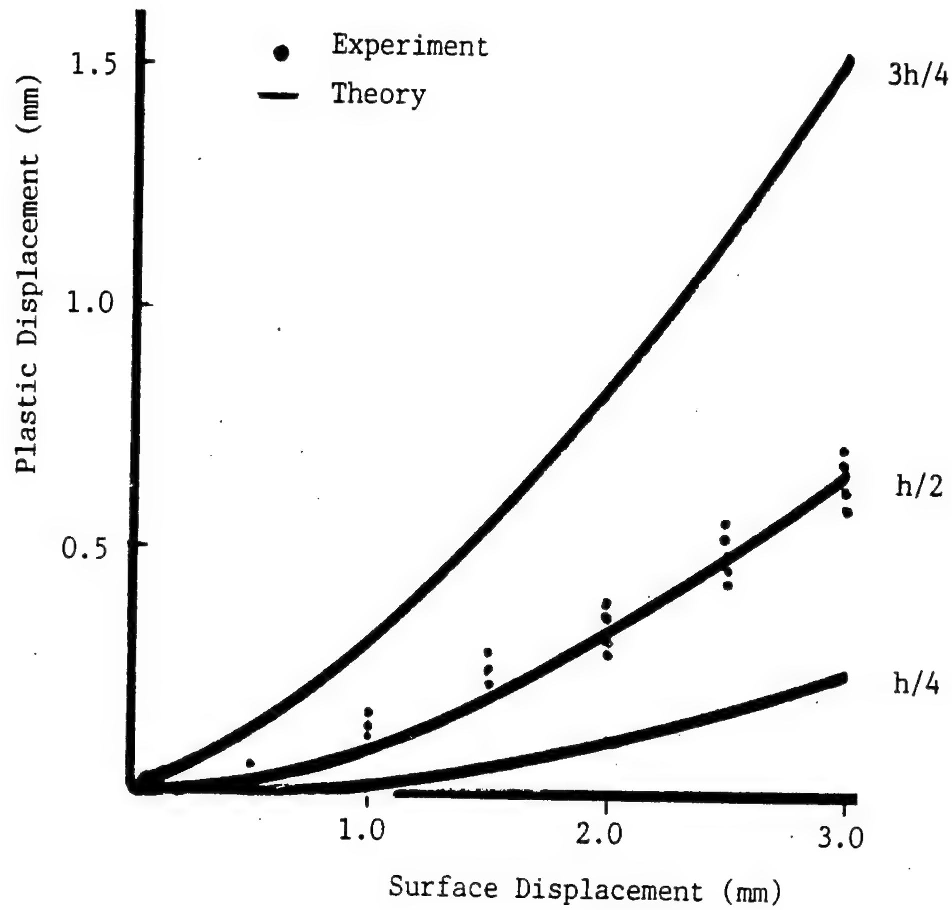
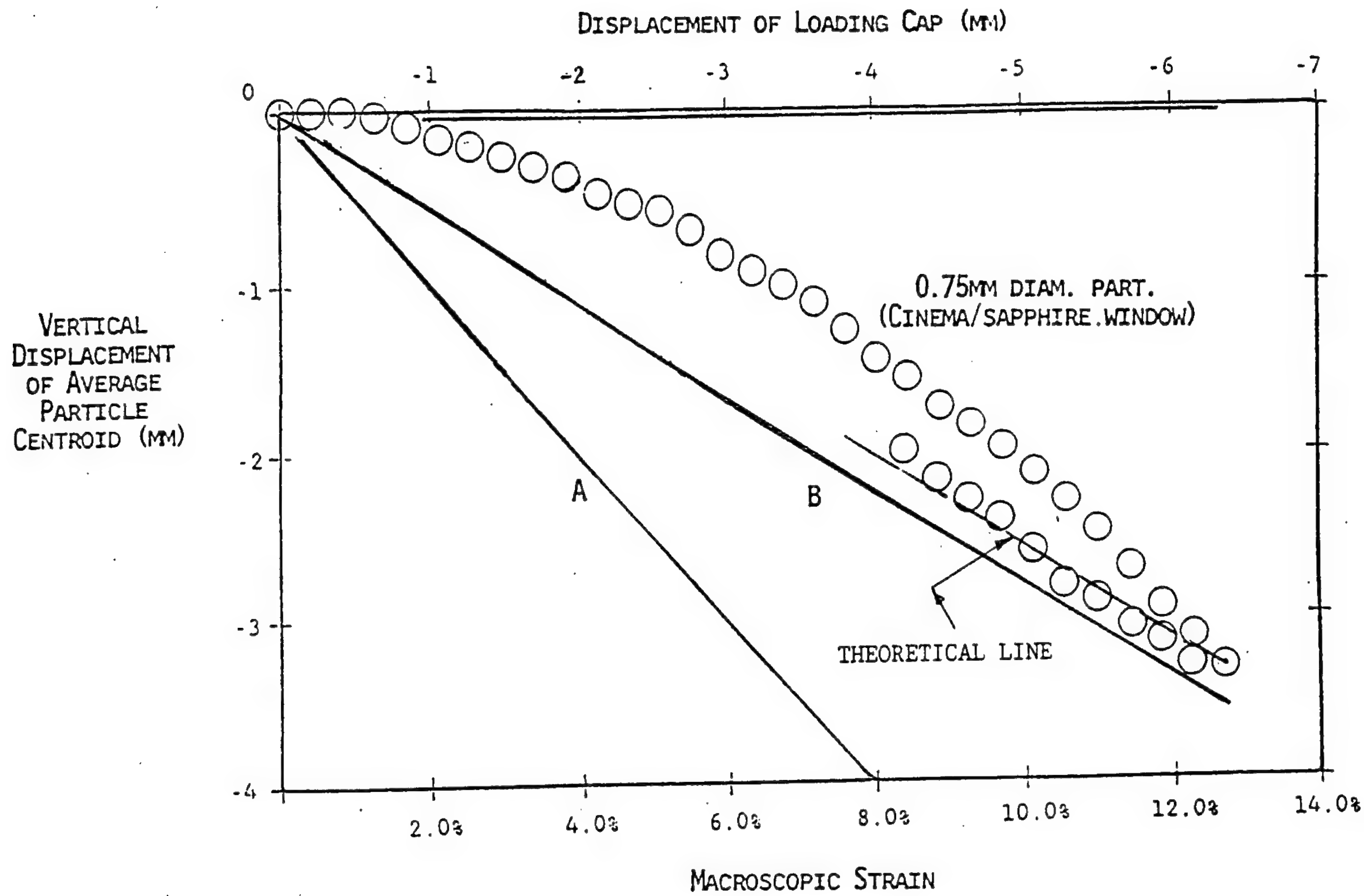


Fig.3. Plastic Displacement vs Surface Displacement at Various Height Locations



DISPLACMENT OF THE AVERAGE PARTICLE CENTROID;
A; ADJACENT TO THE LOADING CAP.
B; AT 0.56H IF A LINEAR DISPLACEMENT DISTRIBUTION IS ASSUMED (LOCAL THEORY).

Fig. 4. Experimental Displacement Points Near the Mid-Point and Comparison With the Theoretical Line

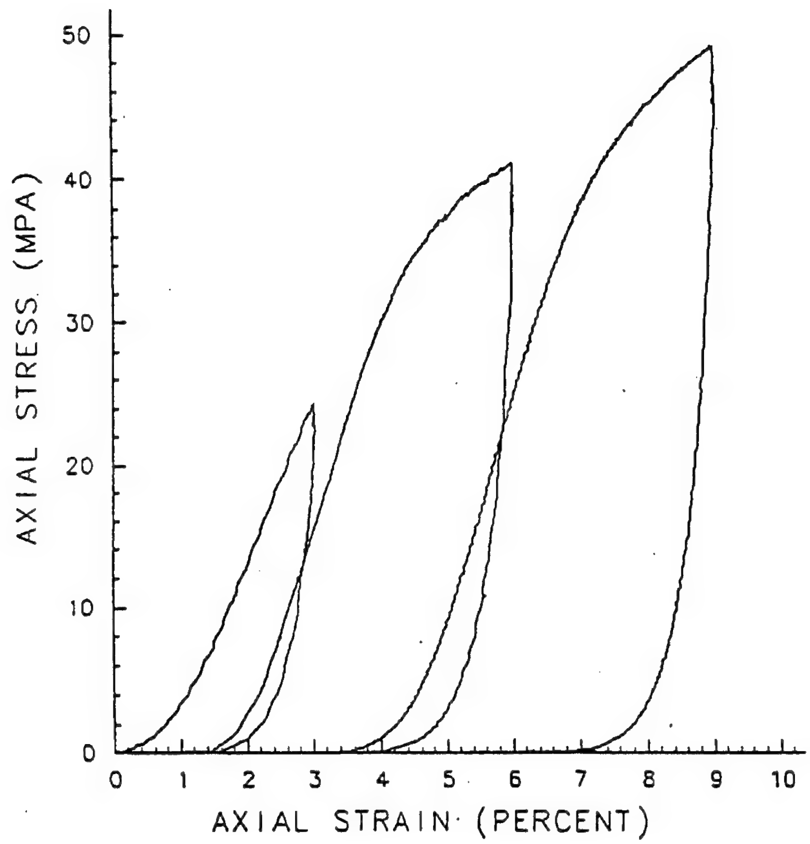


Fig. 5. Experimentally Obtained Stress-Strain Response Under Axial Strain Conditions

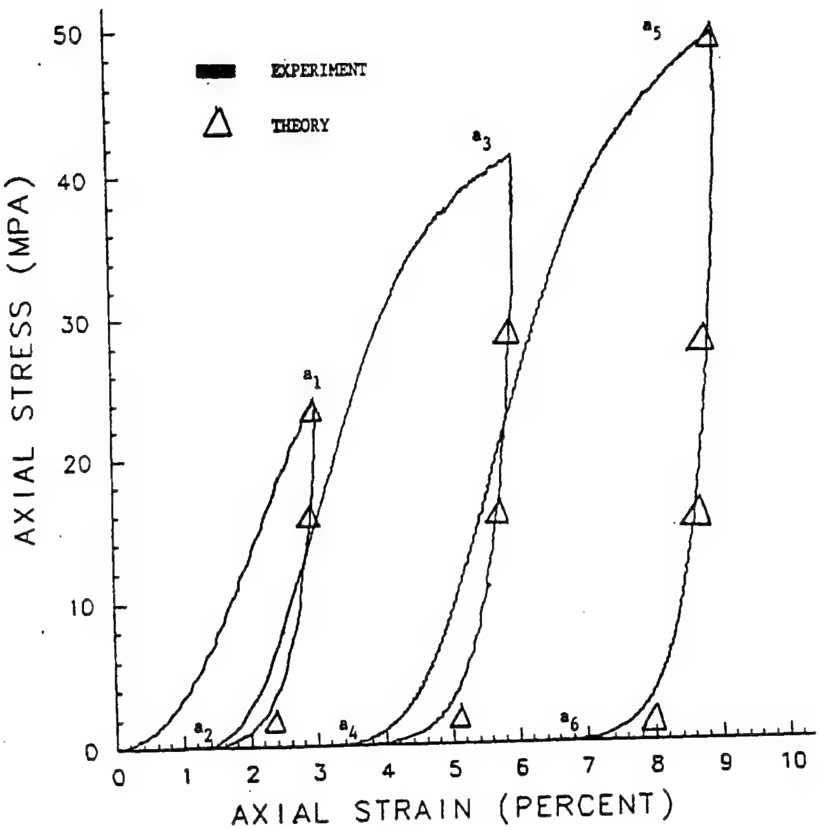


Fig. 6. Unloading Stress-Strain Response Under Axial Strain Conditions

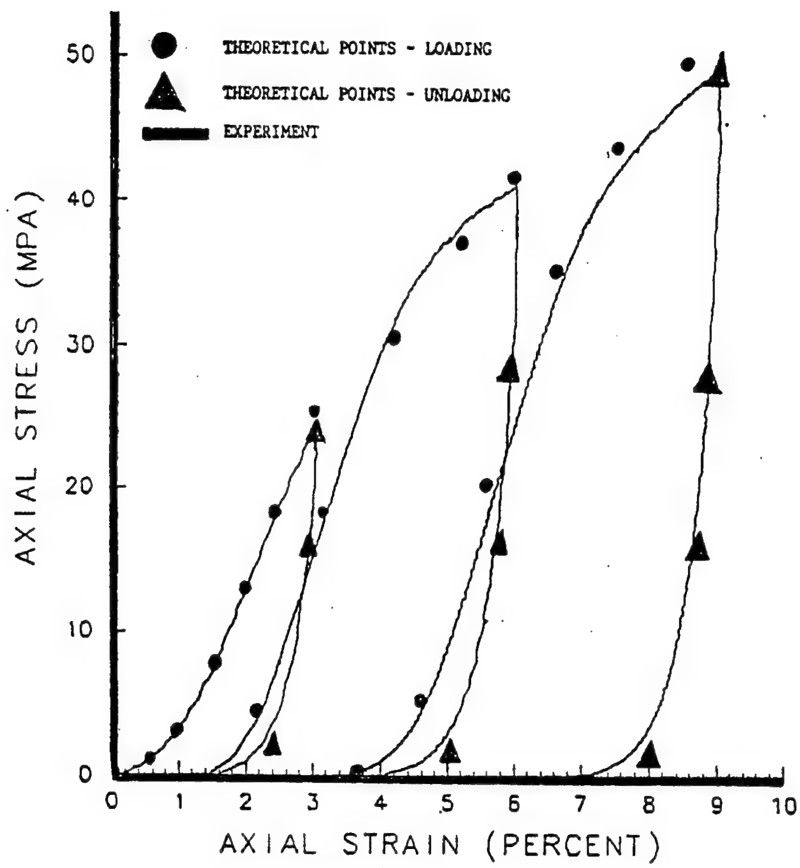


Fig. 7. Experimental and Theoretical Stress-Strain Response Under Axial Strain Conditions

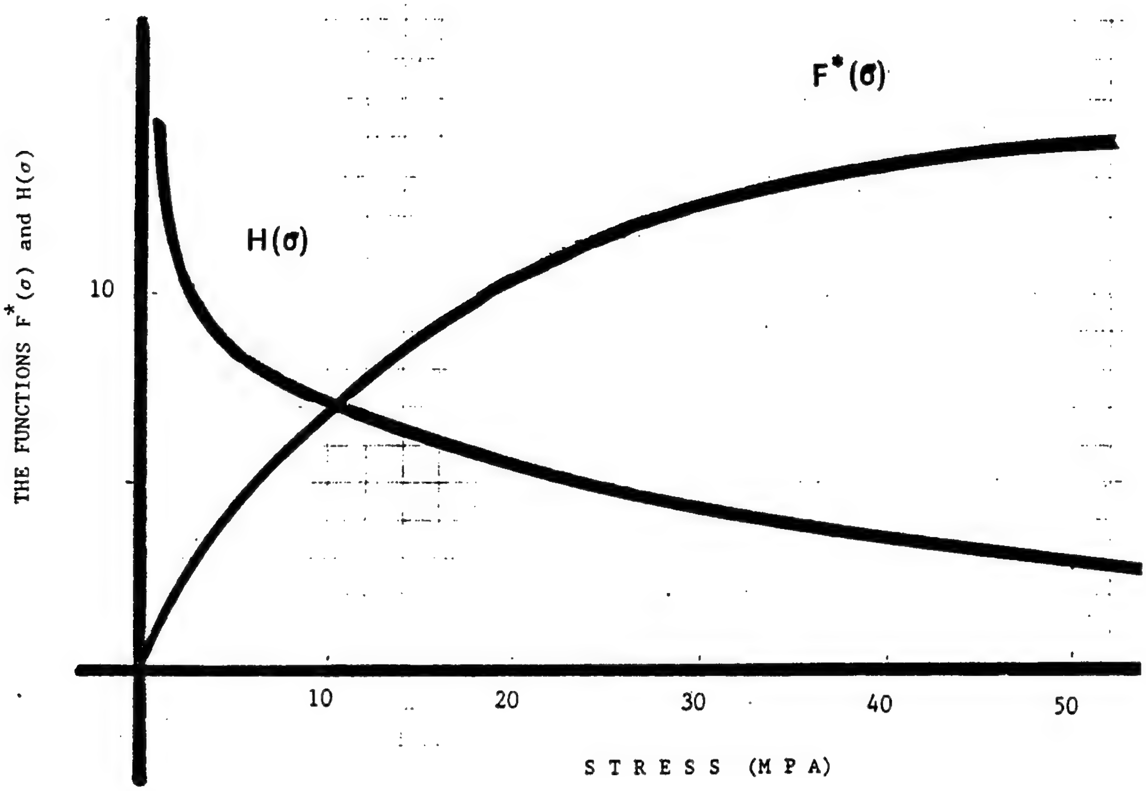


Fig. 8. The Functions F^* and H vs Stress

SECTION 4

DETERMINATION OF MATERIAL CONSTANTS

ONE-DIMENSIONAL HISTORIES

We begin this Section with the method of determination of material constants, and functions in one-dimensional strain histories.

Case (i) Loading Histories

The relevant equation is Eq. (3.29) -- which determines the displacement field in the cuboid -- and its companion equation (3.1a), which we repeat here as Eqs. (4.1 and (4.1a):

$$\frac{a}{h}\frac{\partial^2 u^p}{\partial x^2} = \frac{\partial u^p}{\partial U} \tag{4.1}$$

$$a = \left(\frac{A_o}{b_o}\right)\left(1-\frac{n}{N}\right)^{-1}\left(\frac{1}{N}\right) \tag{4.1a}$$

and the equation for the determination of the stress, i.e.,

$$\sigma = F^*(\sigma) e^{\beta\epsilon}(\epsilon')^{1/2} \tag{4.2}$$

which is Eq. (3.47). We note that in the case of loading histories

$$u^p \doteq u \quad , \quad \epsilon^p = \epsilon \tag{4.3a,b}$$

Case (ii). Unloading Histories

The relevant equations here are:

$$\frac{\partial^2 u^e}{\partial x^2} = 0 \quad ; \quad u = u^e \tag{4.4}$$

$$\sigma = \sigma(\epsilon^e) \tag{4.5}$$

Method of Determination of the Constants and Functions

Case (i). Loading Histories

The constants and functions to be determined are the constant "a" in Eq. (4.1), the constant β in Eq. (4.2) an the function $F^*(\sigma)$ in Eq. (4.2).

The Constant "a". Given a granular medium in the form of a cuboid it will not be known, *ab initio*, that the solution of Eq. (4.1) can be obtained by an infinite length approximation, because this will depend on the actual height h of the cuboid, and the friction coefficient b_o as well as the elastic constant A_o, in accordance with Eq (4.1a). Thus the closed form solution given in Eq. (3.25) cannot be used.

The approach, therefore, is to solve Eq. (4.1) by a finite difference technique with a three node approximation, one node at x = 0, one at x = h/2 and one at x = h. The boundary conditions are such that $u^p(o) = 0$, $u^p(h/2) \equiv u^p$ and $u^p(h) = U$. The solution is given by Eq. (3.27), i.e.,

$$u^p = \frac{1}{2}\left[U - \frac{1}{c}\left(1 - e^{-cU}\right)\right] \tag{4.6}$$

where $c = 8a/h^3$.

Thus if u^p is measured at or close to the midpoint of the cuboid the constant c can be determined by matching the right-hand side of Eq. (4.6) to the measured relation between u^p and U as in Figure 9. The value of c thus found as 0.6.

With this value of c at hand, Eq. (4.1) was then solved more accurately using five nodes in the finite difference scheme, as discussed in Section 3. The solution in that section was given in terms of values of $c^*(= 2\ c)$ equal to 0.9, 1 and 1.2, and was determined that $c^* = 1$ was the optimal value of c^* (i.e., $c = 0.5$) for the most accurate theoretical representation of the observed data. Since $c^* = 16\ a/h^3$ the constant "a" was thus determined.

The Constant β

In Figure 10 we highlight the loading stress-strain curves oa_1, a_2a_3, a_4a_5. The horizontal line ABC is a constant stress line that intersects the curves oa_1 at A, a_2a_3 at B and a_4a_5 at C. Thus $\sigma_A = \sigma_B = \sigma_C$,

In reference to Eq. (4.2) we note the following value of ϵ_c in units of 1 percent (0.01):

$$\epsilon(A) = 2.2 \quad , \quad ,\epsilon(B) = 3.0 \quad , \quad \epsilon(C) = (5.4).$$

We also note that

$$\epsilon'(A) = 2.2 \quad , \quad ,\epsilon'(B) = 3.0 \quad , \quad \epsilon'(C) = (1.6).$$

In view of Eq. (4.2), we have the following relations:

$$\sigma_A = F^*(\sigma_A)e^{\beta\epsilon(A)}(\epsilon'_A)^{1/2} \tag{4.7a}$$

$$\sigma B = F^*(\sigma_B)e^{\beta\epsilon(B)}(\epsilon'_B)^{1/2} \tag{4.7b}$$

$$\sigma_C = F^*(\sigma_C)e^{\beta\epsilon(C)}(\epsilon'_C)^{1/2} \tag{4.7c}$$

Thus it follows that, since $\sigma_A = \sigma_B = \sigma_C$,

$$e^{\beta \in (A)} (\in'_A)^{1/2} = e^{\beta \in (B)} (\in'_B)^{1/2} = e^{\beta \in (C)} (\in'_C)^{1/2} \qquad (4.8a,b,c)$$

Hence typically

$$\beta(\in (B) - \in (A)) = \frac{1}{2} \log\left(\frac{\in (A)}{\in (B)} \right) . \qquad (4.9)$$

Six values of β for six such computations in accordance with the above procedure are shown:

$$\beta = \{0.046, 0.050, 0.060, 0.060, 0063, 0.066\}$$

An average β was then set at 0.060.

The Function F*(σ)

The function F* was determined directly for an Eq. (4.2) and the loading curve a_4a_5 of Figure 10. Evidently,

$$F^*(\sigma) = \sigma e^{-\beta \in} (\in')^{1/2} \qquad (4.10)$$

where $\in$ and $\in'$ are measured in units of 1 percent.

Since every term on the right-hand side of Eq. (4.10) is known at every point on the curve a_4a_5, then F*(σ) can be determined for all σ on that curve. The function F*(σ) so determined is shown in Figure 10. Also shown in that Figure is the function H(σ) where

$$H(\sigma) = \frac{\sigma}{F^*(\sigma)} \qquad (4.11)$$

Hence,

$$H(\sigma) = e^{\beta \in} (\in')^{1/2} \qquad (4.12)$$

and therefore,

$$\sigma = H^{-1}\left(e^{\beta\epsilon}(\epsilon')^{1/2}\right) \tag{4.13}$$

The importance of Eq. (4.13) is that it is an explicit equation for the stress for all loading curves, emanating at $\sigma = 0$. In fact this equation was used to predict the stress response for the curves oa, and a_2a_3, a result already shown in Figure 8.

Determination of Material Constants ad Function in Three Dimensions

1. The Mobile Phase

The three-dimensional motion of the mobile phase is given by Eqs. (2.54) and (2.57), i.e.,

$$\mu\, u^*_{i,jj} = b\frac{\partial u^*_i}{\partial z} \tag{4.14}$$

and

$$(\lambda + 2\mu)\phi_{,ii} = b\frac{\partial \phi}{\partial z} \tag{4.15}$$

Recall that the displacement field u_i is given by Eq. (4.16):

$$u^p_i = u^*_i + \bar{u}_i \tag{4.16}$$

where u^*_i is an isochronic field, i.e., $u^*_{i,j} = 0$ while $\bar{u}_i$ is irrotational, i.e., $\bar{u}_{i,j} = u_{j,i}$, and:

$$\bar{u}_i = \phi_{,i} \tag{4.17}$$

The Irrotational Field

In the uniaxial experiment discussed in Section 3,

$$u_1^p = u^p(x_1)\,,\; u_2^p = 0\,,\; u_3^p = 0 \quad .$$

Thus

$$u_{1,2}^p = u_{2,1}^p\,,\; u_{1,3}^p = u_{2,3}^p = u_{3,2}^p \quad .$$

and the field is irrotational.

In this, uniaxial, case Eq. (2.55) may be applied directly. Thus

$$(\lambda + 2\mu)\bar{u}_{xx} = b\frac{\partial \bar{u}}{\partial z} \tag{4.18}$$

where we have set $x_1 \equiv x$. Thus, in Eq. (2.41)

$$A = \lambda + 2\mu \tag{4.19}$$

and

$$a = \frac{\lambda + 2\mu}{b} = \bar{a} \tag{4.20}$$

as in Eqs. (3.32d) and (3.35).

Also,

$$A = A_o F(\sigma) e^{\beta\epsilon} \tag{4.21}$$

$$b = b_o F(\sigma) e^{\beta\epsilon} \tag{4.22}$$

Thus the uniaxial test determines the constants and functions of the irrotational motion, as discussed earlier in this section.

The Isochronic Field

Here the appropriate experiment is a shear experiment such as the one conducted in a "sheer box". See Ref. 7 for instance. Of note is the fact that σ is common to both the isochronic and irrotational fields. Thus the function $F(\sigma)$ and the constant β are determined from the irrotational test. There remains the constant a^* where

$$a^* = \frac{b}{\mu} \tag{4.23}$$

and governs the isochronic motion given by Eq. (4.14).

The procedure for determining a^* is precisely the same as that for determining $a(\bar{a})$ in Section 3, by measuring the displacement at the half-height of the shear box as a function of the surface displacement and then using Eq. (4.6).

THE ELASTIC PHASE

We proceed on the assumption, normally made in soil mechanics, that the elastic shear response is linear while the hydrostatic response is non-linear. Thus

$$s_{ij} = 2\mu\, e^e_{ij} \tag{4.24}$$

$$\sigma = K\left(\epsilon^e\right) \tag{4.25}$$

where K is a function of the elastic hydrostatic strain.

The Shear Modulus μ

Under pure sheer unloading conditions let s be the shear stress and e^e the corresponding elastic shear strain. Then

$$2\mu = s/e^e \tag{4.26}$$

and hence μ is found.

The Function $K(\in^e)$

This may be found by means of the uniaxial strain test under unloading conditions. We note that, in view of Eq. (4.24)

$$\sigma_1 - \sigma_2 = 2\mu \in_1 \tag{4.27}$$

since $\in_2 = 0$. Also in view of Eq. (4.25)

$$(\sigma_1 + 2\sigma_2) = 3K\left(\frac{\in_1}{3}\right) = K^*(\in_1) \tag{4.28}$$

Eliminating σ_2 from Eqs. (4.27) and (4.28);

$$\sigma_1 = \frac{4}{3}\mu \in_1 + K^*(\in_1) \tag{4.29}$$

Hence K* and thus $K(\in)$ may thus be found.

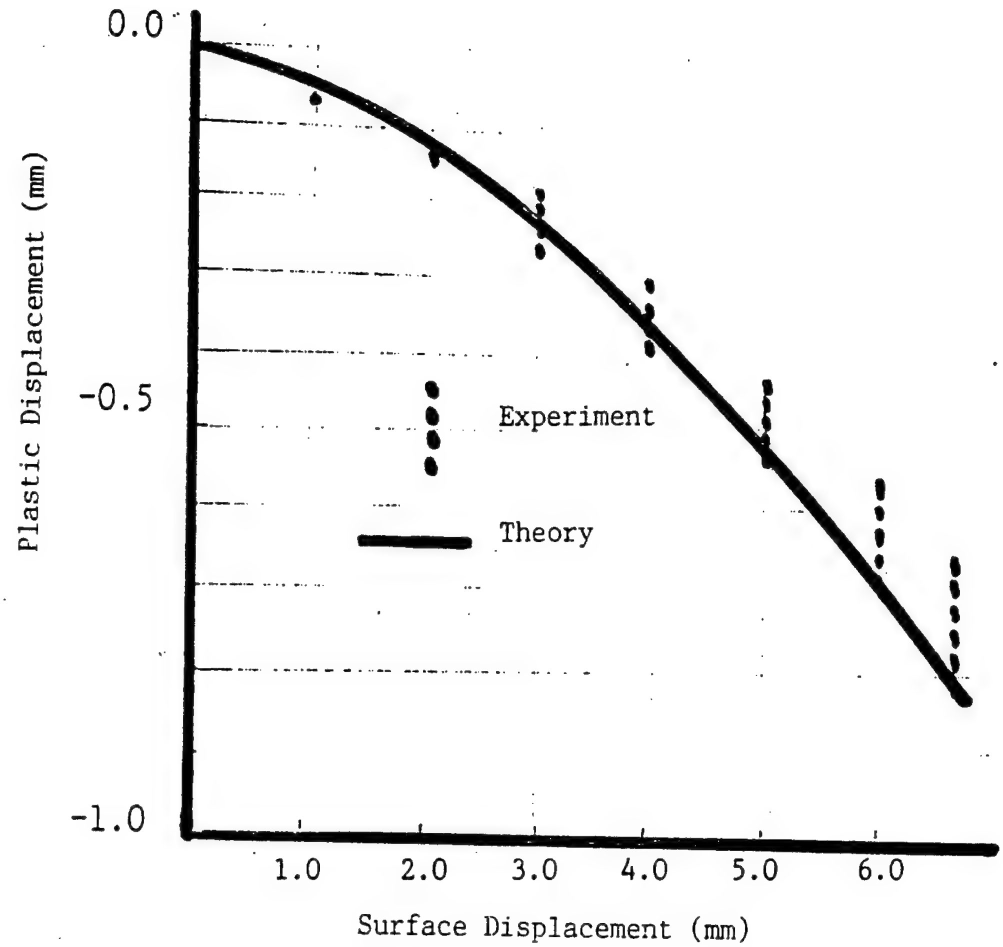


Fig. 9. Plastic Displacement vs Surface Displacement at Mid-Point

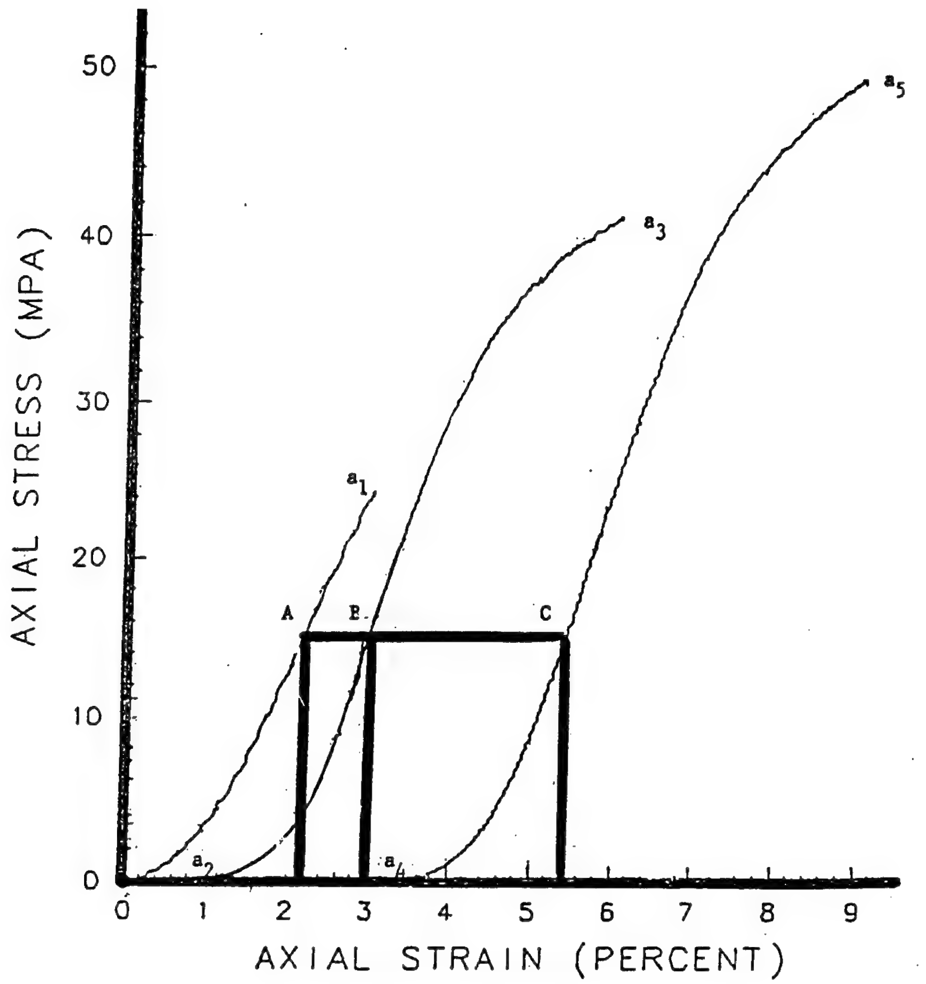


Fig. 10. Geometric Determination of the Parameter β

SECTION 5

DISCUSSION

There is ample evidence, both experimental [6] and theoretical, (as discussed by the author in past work [1] and in the current study and by other authors [7]) to the effect that the mechanical response of granular domains, under surface tractions, is non-local. The precise nature of the non-local behavior is a matter for further research.

In this study and in the work that preceded it [1-3] we have identified three causes for non-locality: (a) internal friction giving rise to non-local thermodynamic formulations, (b) configurational entropy and temperature giving rise to diffusive motion of the "thermal" kind and (c) mobility which is most likely to be a field with its own field equation.

RECOMMENDATIONS

Mindful of the need to lend practical value to constitutive equations, further research is needed to identify the practical circumstances where a local theory will suffice, but also reveal "pathological circumstances", such as banding and instability, where clearly a non-local theory is needed for the physical understanding and prediction of such pathological events.

Continued research in the study of non-local effects in the motion of granular domains is therefore strongly recommended.

SECTION 6

LIST OF REFERENCES

1. Valanis, K. C., "A Micromechanical Foundation of Non-Local Behavior of Granular Media", ENDOCHRONICS Final Report to University of New Mexico Research Institute, P.O.N. 831741, June 1990.

2. Valanis, K. C., and J. J. Gill, "A Micromechanical Foundation of Plastic Behavior of Granular Media", MICROMECHANICS OF FAILURE OF QUASI-BRITTLE MATERIALS, Eds. S. P. Shah, *et al.*, Elsevier Applied Science (1990).

3. Valanis, K. C., and J. J. Gill, "A Micromechanical Theory of Non-Local Behavior of Granular Media," COMPUTER METHODS AND ADVANCES IN GEOMECHANICS, A. A., Balkema (1991).

4. Gill, J. J., "An Experimental Program for the Response of a Granular Cuboid to a Variety of Surface Displacement Histories", Kirtand AFB Communication: GJJ 1 - Private Correspondence (1991).

5. Trent, B., "Recent Advances in Visualization Based on 2D Discrete Element Modeling Test Results," Presented at MICROMECHANICS OF GRANULAR MEDIA, by Major John J. Gill, Kirtland AFB (1991).

6. Gill, J. J. *et al.*, "Measurement of the Micromechanical Response of Lightly Cemented Granular Soils under Uniaxial Strain Conditions", MICROMECHANICS OF FAILURE OF QUASI-BRITTLE MATERIALS, Elsevier Applied Science (1990).

7. Mogami, T., "Selected Papers", Gihodo Publishing Co., Tokyo, Japan (1974).

8. Biot, M. A., "Thermoelasticity and Irreversible Thermodynamics", *J. App. Phys.*, 27, 240, (1954).

9. Valanis, K. C., "A Theory of Viscoplasticity Without a Yield Surface", Part I - General Theory, *Archives of Mechanics*, 23, 517-533 (1971).

10. Valanis, K. C., "Fundamental Consequences of a New Intrinsic Time Measure -- Plasticity as a Limit of the Endochronic Theory," *Archives of Mechanics*, 32, 171-191 (1980).

11. Born, M., "Natural Philosophy of Cause and Chance," Dover Publications, NY (1964).

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